$A - B$ (the difference of A and B): the set containing those elements that are in A but not in B

$\overline{A}$ (the complement of A): the set of elements in the universal set that are not in A

$A \oplus B$ (the symmetric difference of A and B): the set containing those elements in exactly one of A and B

membership table: a table displaying the membership of elements in sets

function from A to B: an assignment of exactly one element of B to each element of A

domain of f: the set A, where f is a function from A to B

codomain of f: the set B, where f is a function from A to B

b is the image of a under f: $b = f(a)$

a is a pre-image of b under f: $f(a) = b$

range of f: the set of images of f

onto function, surjection: a function from A to B such that every element of B is the image of some element in A

one-to-one function, injection: a function such that the images of elements in its domain are distinct

one-to-one correspondence, bijection: a function that is both one-to-one and onto

inverse of f: the function that reverses the correspondence given by f (when f is a bijection)

$f \circ g$ (composition of f and g): the function that assigns $f(g(x))$ to x

$\lfloor x \rfloor$ (floor function): the largest integer not exceeding x

$\lceil x \rceil$ (ceiling function): the smallest integer greater than or equal to x

partial function: an assignment to each element in a subset of the domain a unique element in the codomain

sequence: a function with domain that is a subset of the set of integers

geometric progression: a sequence of the form $a, ar, ar^2, \ldots,$ where a and r are real numbers

arithmetic progression: a sequence of the form a, $a + d$, $a + 2d, \ldots,$ where a and d are real numbers

string: a finite sequence

empty string: a string of length zero

recurrence relation: a equation that expresses the nth term a_n of a sequence in terms of one or more of the previous terms of the sequence for all integers n greater than a particular integer

$\sum_{i=1}^{n} a_i$: the sum $a_1 + a_2 + \cdots + a_n$

$\prod_{i=1}^{n} a_i$: the product $a_1 a_2 \cdots a_n$

cardinality: two sets A and B have the same cardinality if there is a one-to-one correspondence from A to B

countable set: a set that either is finite or can be placed in one-to-one correspondence with the set of positive integers

uncountable set: a set that is not countable

$\aleph_0$ (aleph null): the cardinality of a countable set

$\mathfrak{c}$: the cardinality of the set of real numbers

Cantor diagonalization argument: a proof technique used to show that the set of real numbers is uncountable

computable function: a function for which there is a computer program in some programming language that finds its values

uncomputable function: a function for which no computer program in a programming language exists that finds its values

continuum hypothesis: the statement there no set A exists such that $\aleph_0 < |A| < \mathfrak{c}$

matrix: a rectangular array of numbers

matrix addition: see page 178

matrix multiplication: see page 179

$\mathbf{I}_n$ (identity matrix of order n): the $n \times n$ matrix that has entries equal to 1 on its diagonal and 0s elsewhere

$\mathbf{A}^t$ (transpose of $\mathbf{A}$): the matrix obtained from $\mathbf{A}$ by interchanging the rows and columns

symmetric matrix: a matrix is symmetric if it equals its transpose

zero–one matrix: a matrix with each entry equal to either 0 or 1

$\mathbf{A} \vee \mathbf{B}$ (the join of A and B): see page 181

$\mathbf{A} \wedge \mathbf{B}$ (the meet of A and B): see page 181

$\mathbf{A} \odot \mathbf{B}$ (the Boolean product of A and B): see page 182

RESULTS

The set identities given in Table 1 in Section 2.2
The summation formulae in Table 2 in Section 2.4
The set of rational numbers is countable.
The set of real numbers is uncountable.

Review Questions

1. Explain what it means for one set to be a subset of another set. How do you prove that one set is a subset of another set?

2. What is the empty set? Show that the empty set is a subset of every set.

3. **a)** Define $|S|$, the cardinality of the set S.
b) Give a formula for $|A \cup B|$, where A and B are sets.

4. **a)** Define the power set of a set S.
b) When is the empty set in the power set of a set S?
c) How many elements does the power set of a set S with n elements have?

5. **a)** Define the union, intersection, difference, and symmetric difference of two sets.
b) What are the union, intersection, difference, and symmetric difference of the set of positive integers and the set of odd integers?

6. **a)** Explain what it means for two sets to be equal.
b) Describe as many of the ways as you can to show that two sets are equal.
c) Show in at least two different ways that the sets $A - (B \cap C)$ and $(A - B) \cup (A - C)$ are equal.

7. Explain the relationship between logical equivalences and set identities.
8. a) Define the domain, codomain, and range of a function.
   b) Let $f(n)$ be the function from the set of integers to the set of integers such that $f(n) = n^2 + 1$. What are the domain, codomain, and range of this function?
9. a) Define what it means for a function from the set of positive integers to the set of positive integers to be one-to-one.
   b) Define what it means for a function from the set of positive integers to the set of positive integers to be onto.
   c) Give an example of a function from the set of positive integers to the set of positive integers that is both one-to-one and onto.
   d) Give an example of a function from the set of positive integers to the set of positive integers that is one-to-one but not onto.
   e) Give an example of a function from the set of positive integers to the set of positive integers that is not one-to-one but is onto.
   f) Give an example of a function from the set of positive integers to the set of positive integers that is neither one-to-one nor onto.
10. a) Define the inverse of a function.
    b) When does a function have an inverse?
    c) Does the function $f(n) = 10 - n$ from the set of integers to the set of integers have an inverse? If so, what is it?
11. a) Define the floor and ceiling functions from the set of real numbers to the set of integers.
    b) For which real numbers x is it true that $\lfloor x \rfloor = \lceil x \rceil$?
12. Conjecture a formula for the terms of the sequence that begins 8, 14, 32, 86, 248 and find the next three terms of your sequence.
13. Suppose that $a_n = a_{n-1} - 5$ for $n = 1, 2, \ldots$. Find a formula for a_n.
14. What is the sum of the terms of the geometric progression $a + ar + \cdots + ar^n$ when $r \neq 1$?
15. Show that the set of odd integers is countable.
16. Give an example of an uncountable set.
17. Define the product of two matrices **A** and **B**. When is this product defined?
18. Show that matrix multiplication is not commutative.

Supplementary Exercises

1. Let A be the set of English words that contain the letter x, and let B be the set of English words that contain the letter q. Express each of these sets as a combination of A and B.
   a) The set of English words that do not contain the letter x.
   b) The set of English words that contain both an x and a q.
   c) The set of English words that contain an x but not a q.
   d) The set of English words that do not contain either an x or a q.
   e) The set of English words that contain an x or a q, but not both.
2. Show that if A is a subset of B, then the power set of A is a subset of the power set of B.
3. Suppose that A and B are sets such that the power set of A is a subset of the power set of B. Does it follow that A is a subset of B?
4. Let **E** denote the set of even integers and **O** denote the set of odd integers. As usual, let **Z** denote the set of all integers. Determine each of these sets.

   a) $\mathbf{E} \cup \mathbf{O}$ b) $\mathbf{E} \cap \mathbf{O}$ c) $\mathbf{Z} - \mathbf{E}$ d) $\mathbf{Z} - \mathbf{O}$
5. Show that if A and B are sets, then $A - (A - B) = A \cap B$.
6. Let A and B be sets. Show that $A \subseteq B$ if and only if $A \cap B = A$.
7. Let A, B, and C be sets. Show that $(A - B) - C$ is not necessarily equal to $A - (B - C)$.
8. Suppose that A, B, and C are sets. Prove or disprove that $(A - B) - C = (A - C) - B$.
9. Suppose that A, B, C, and D are sets. Prove or disprove that $(A - B) - (C - D) = (A - C) - (B - D)$.
10. Show that if A and B are finite sets, then $|A \cap B| \leq |A \cup B|$. Determine when this relationship is an equality.
11. Let A and B be sets in a finite universal set U. List the following in order of increasing size.
    a) $|A|, |A \cup B|, |A \cap B|, |U|, |\emptyset|$
    b) $|A - B|, |A \oplus B|, |A| + |B|, |A \cup B|, |\emptyset|$
12. Let A and B be subsets of the finite universal set U. Show that $|\overline{A} \cap \overline{B}| = |U| - |A| - |B| + |A \cap B|$.
13. Let f and g be functions from $\{1, 2, 3, 4\}$ to $\{a, b, c, d\}$ and from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$, respectively, with $f(1) = d$, $f(2) = c$, $f(3) = a$, and $f(4) = b$, and $g(a) = 2$, $g(b) = 1$, $g(c) = 3$, and $g(d) = 2$.
    a) Is f one-to-one? Is g one-to-one?
    b) Is f onto? Is g onto?
    c) Does either f or g have an inverse? If so, find this inverse.
14. Suppose that f is a function from A to B where A and B are finite sets. Explain why $|f(S)| \leq |S|$ for all subsets S of A.

15. Suppose that f is a function from A to B where A and B are finite sets. Explain why $|f(S)| = |S|$ for all subsets S of A if and only if f is one-to-one.

Suppose that f is a function from A to B. We define the function S_f from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ by the rule $S_f(X) = f(X)$ for each subset X of A. Similarly, we define the function $S_{f^{-1}}$ from $\mathcal{P}(B)$ to $\mathcal{P}(A)$ by the rule $S_{f^{-1}}(Y) = f^{-1}(Y)$ for each subset Y of B. Here, we are using Definition 4, and the definition of the inverse image of a set found in the preamble to Exercise 42, both in Section 2.3.

***16.** Suppose that f is a function from the set A to the set B. Prove that

- **a)** if f is one-to-one, then S_f is a one-to-one function from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.
- **b)** if f is onto function, then S_f is an onto function from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.
- **c)** if f is onto function, then $S_{f^{-1}}$ is a one-to-one function from $\mathcal{P}(B)$ to $\mathcal{P}(A)$.
- **d)** if f is one-to-one, then $S_{f^{-1}}$ is an onto function from $\mathcal{P}(B)$ to $\mathcal{P}(A)$.
- **e)** if f is a one-to-one correspondence, then S_f is a one-to-one correspondence from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ and $S_{f^{-1}}$ is a one-to-one correspondence from $\mathcal{P}(B)$ to $\mathcal{P}(A)$. [*Hint:* Use parts (a)-(d).]

17. Prove that if f and g are functions from A to B and $S_f = S_g$ (using the definition in the preamble to Exercise 16), then $f(x) = g(x)$ for all $x \in A$.

18. Show that if n is an integer, then $n = \lceil n/2 \rceil + \lfloor n/2 \rfloor$.

19. For which real numbers x and y is it true that $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$?

20. For which real numbers x and y is it true that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$?

21. For which real numbers x and y is it true that $\lceil x + y \rceil = \lceil x \rceil + \lfloor y \rfloor$?

22. Prove that $\lfloor n/2 \rfloor \lceil n/2 \rceil = \lfloor n^2/4 \rfloor$ for all integers n.

23. Prove that if m is an integer, then $\lfloor x \rfloor + \lfloor m - x \rfloor = m - 1$, unless x is an integer, in which case, it equals m.

24. Prove that if x is a real number, then $\lfloor \lfloor x/2 \rfloor /2 \rfloor = \lfloor x/4 \rfloor$.

25. Prove that if n is an odd integer, then $\lceil n^2/4 \rceil = (n^2 + 3)/4$.

26. Prove that if m and n are positive integers and x is a real number, then

$$\left\lfloor \frac{\lfloor x \rfloor + n}{m} \right\rfloor = \left\lfloor \frac{x + n}{m} \right\rfloor.$$

***27.** Prove that if m is a positive integer and x is a real number, then

$$\lfloor mx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{m} \right\rfloor + \left\lfloor x + \frac{2}{m} \right\rfloor + \cdots + \left\lfloor x + \frac{m-1}{m} \right\rfloor.$$

***28.** We define the **Ulam numbers** by setting $u_1 = 1$ and $u_2 = 2$. Furthermore, after determining whether the integers less than n are Ulam numbers, we set n equal to the next Ulam number if it can be written uniquely as the sum of two different Ulam numbers. Note that $u_3 = 3$, $u_4 = 4$, $u_5 = 6$, and $u_6 = 8$.

- **a)** Find the first 20 Ulam numbers.
- **b)** Prove that there are infinitely many Ulam numbers.

29. Determine the value of $\prod_{k=1}^{100} \frac{k+1}{k}$. (The notation used here for products is defined in the preamble to Exercise 43 in Section 2.4.)

***30.** Determine a rule for generating the terms of the sequence that begins 1, 3, 4, 8, 15, 27, 50, 92, . . . , and find the next four terms of the sequence.

***31.** Determine a rule for generating the terms of the sequence that begins 2, 3, 3, 5, 10, 13, 39, 43, 172, 177, 885, 891, . . . , and find the next four terms of the sequence.

32. Show that the set of irrational numbers is an uncountable set.

33. Show that the set S is a countable set if there is a function f from S to the positive integers such that $f^{-1}(j)$ is countable whenever j is a positive integer.

34. Show that the set of all finite subsets of the set of positive integers is a countable set.

****35.** Show that $|\mathbf{R} \times \mathbf{R}| = |\mathbf{R}|$. [*Hint:* Use the Schröder-Bernstein theorem to show that $|(0, 1) \times (0, 1)| = |(0, 1)|$. To construct an injection from $(0, 1) \times (0, 1)$ to $(0, 1)$, suppose that $(x, y) \in (0, 1) \times (0, 1)$. Map (x, y) to the number with decimal expansion formed by alternating between the digits in the decimal expansions of x and y, which do not end with an infinite string of 9s.]

****36.** Show that $\mathbf{C}$, the set of complex numbers has the same cardinality as $\mathbf{R}$, the set of real numbers.

37. Find $\mathbf{A}^n$ if $\mathbf{A}$ is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

38. Show that if $\mathbf{A} = c\mathbf{I}$, where c is a real number and $\mathbf{I}$ is the $n \times n$ identity matrix, then $\mathbf{AB} = \mathbf{BA}$ whenever $\mathbf{B}$ is an $n \times n$ matrix.

39. Show that if $\mathbf{A}$ is a 2×2 matrix such that $\mathbf{AB} = \mathbf{BA}$ whenever $\mathbf{B}$ is a 2×2 matrix, then $\mathbf{A} = c\mathbf{I}$, where c is a real number and $\mathbf{I}$ is the 2×2 identity matrix.

40. Show that if $\mathbf{A}$ and $\mathbf{B}$ are invertible matrices and $\mathbf{AB}$ exists, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

41. Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathbf{0}$ be the $n \times n$ matrix all of whose entries are zero. Show that the following are true.

- **a)** $\mathbf{A} \odot \mathbf{0} = \mathbf{0} \odot \mathbf{A} = \mathbf{0}$
- **b)** $\mathbf{A} \vee \mathbf{0} = \mathbf{0} \vee \mathbf{A} = \mathbf{A}$
- **c)** $\mathbf{A} \wedge \mathbf{0} = \mathbf{0} \wedge \mathbf{A} = \mathbf{0}$